FIRM STRUCTURE AND OCCUPATIONAL SORTING

Fernando Lopes
UW-Madison

Gabriel Toledo
Purdue University

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MOTIVATION

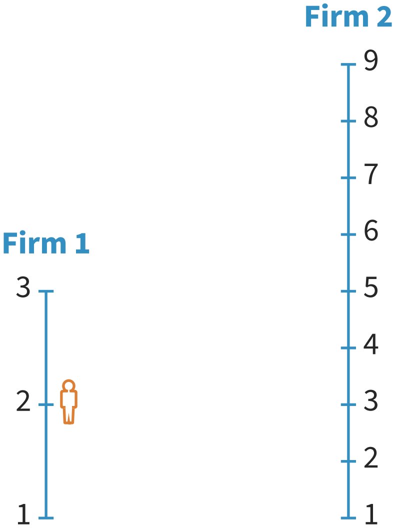
- Allocation of workers across and inside firms is key for productivity
- Workers sort into occupations/hierarchies as well as firms
- However, occupational/hierarchical structure is endogenous to labor market conditions

How does <u>hierarchical structure</u> affect the sorting patterns we expect to see across occupations and firms?

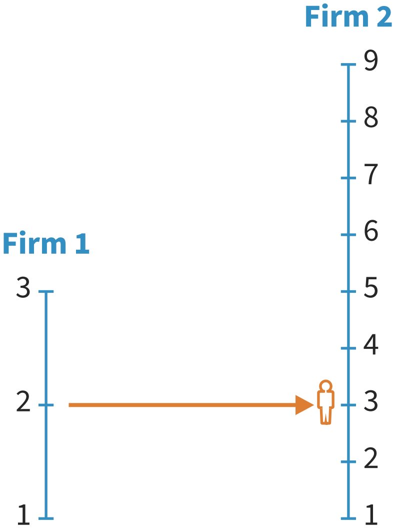
WHY STRUCTURE MATTERS

- Firm structure reflects firm productivity/quality of projects
- By selecting into firms, workers also select into occupations
- We expect to see some correlation between where a person works and their position in a hierarchy
- But how to discipline that?

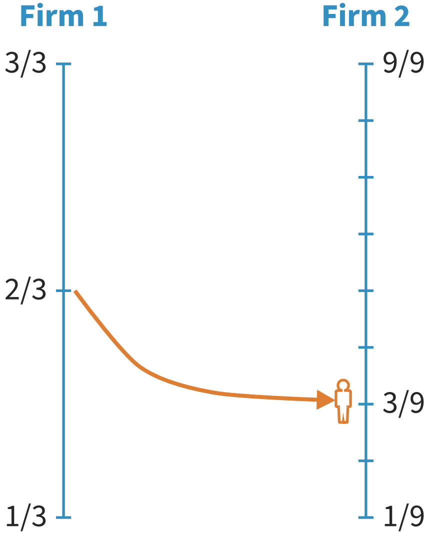
WHY STRUCTURE MATTERS



WHY STRUCTURE MATTERS



WHY STRUCTURE MATTERS



THIS PAPER

- Data - Stylized facts
  - German matched employer-employee data (LIAB); entire composition of firms
  - Job switchers when moving to better firms (higher leave-one out avg wage)
    - ★ Weakly higher number of subordinates
    - ★ Strongly *smaller* relative rank in the hierarchy
- Theory - Model of firm structure and sorting
  - Tractable setting: endogenous production function, heterogeneous workers inside the firm, and nontrivial sorting patterns
  - Comparing hierarchies: NAM across *layers* but PAM across *ranks*
  - Aggregation result that reconciles this model with canonical macro models

LITERATURE AND CONTRIBUTION

- **Sorting with multi-worker firms** (Kremer (1993), Kremer and Maskin (1996), Eeckhout and Kircher (2018), Boerma, Tsyvinski, and Zimin (2021))
  - Endogenous firm size, heterogeneous workers
- **Knowledge hierarchies** (Garicano (2000), Garicano and Rossi-Hansberg (2006))
  - Richer sorting patterns, with NAM among occupations
- **Endogenous firm structure** (Deming (2017), Adenbaum (2022), Freund (2022))

EMPIRICAL MOTIVATION

LIAB - WORKER-ESTABLISHMENT DATA

- Entire biographies of the complete workforce of a sample of establishments
- Organized in a yearly panel 2010-2017 (Dauth and Eppelsheimer (2020)) ◂ Cleaning
- Information on wages, occupation (KldB 2010), establishment identifier, industry, location, etc.
- Focus on West Germany, private sector, full-time workers aged 20-65
- Able to identify job switchers and compare its position in the firm before and after the switch

LAYERS AND RANKS WITHIN FIRMS

- Order workers within each firm-year by real wage
- Define a layer a worker is as number of workers with lower wages (can also work with binned layers)
- Measure of how many "subordinates" a worker has
- Define rank as the relative position in the hierarchy (rank = layer / firm size)
- Measure of how close to the top of the hierarchy a worker is

JOB SWITCHERS

- Focus on workers that switch firms from one year to the next (Jarosch, Oberfield, and Rossi-Hansberg (2021),Gregory (2020))
- Measure of firm quality: leave-one-out average wage
- How does a change in firm quality affect layers and ranks?
- For a worker i moving from firm j to j' at time t:

$$log(\phi_{ij't}) - log(\phi_{ijt-1}) = \beta_0 + \beta_1 \left[log(z_{j't}) - log(z_{jt-1}) \right] + \text{Controls} + \varepsilon_{ijt}$$

$$\phi_{ijt} \in \{\text{Layer}_{ijt}, \text{Rank}_{ijt}\} \text{ and } z_{jt} = \text{quality of firm } j \text{ at} t$$

- Controls: Firm size, age, tenure, occupation, industry, location, year fixed effects

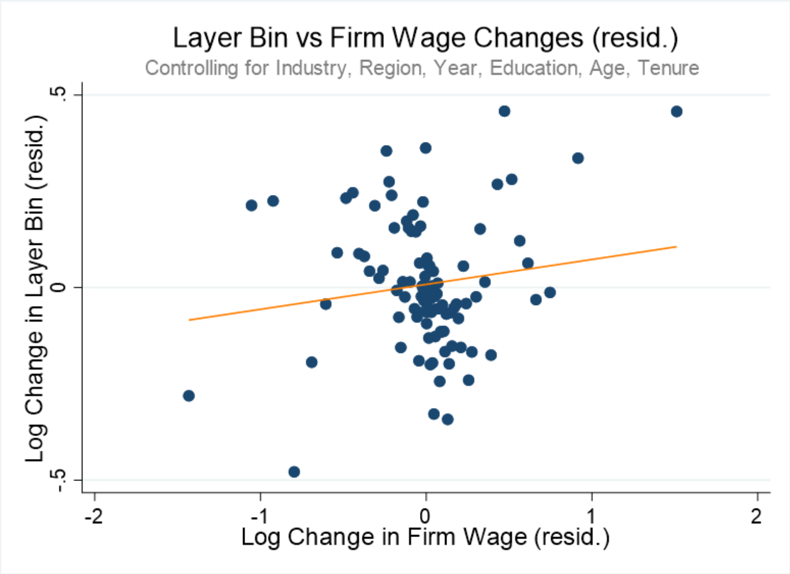
JOB SWITCHERS

Job Switch: Effect of Firm Quality Changes on Rank, Layer, and Layer Bin

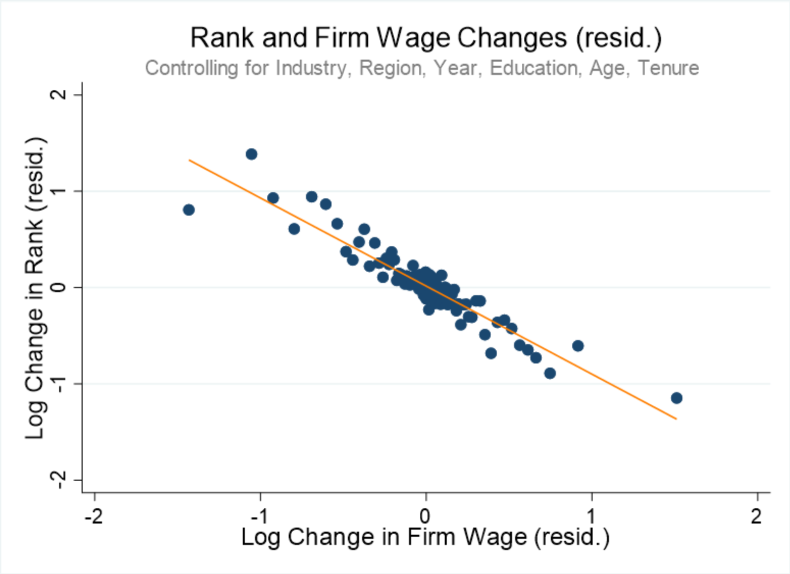
	Layer	Layer Bin	Rank
Log Diff *z*	0.143	0.065	-0.915
(p-value)	(0.471)	(0.473)	(0.000)
N	7,177	7,177	7,177

- Layers: we are rejecting PAM; if anything increase in firm quality leads to weakly a higher layer (NAM)
- Ranks: evidence of PAM: Increase in 10% in firm quality leads to decrease in rank of 9.15%

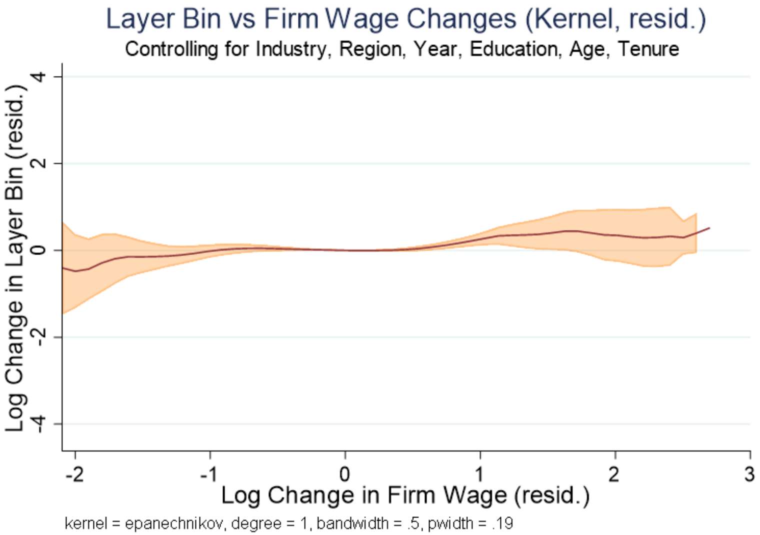
RESIDUALIZED BINSCATTER PLOTS



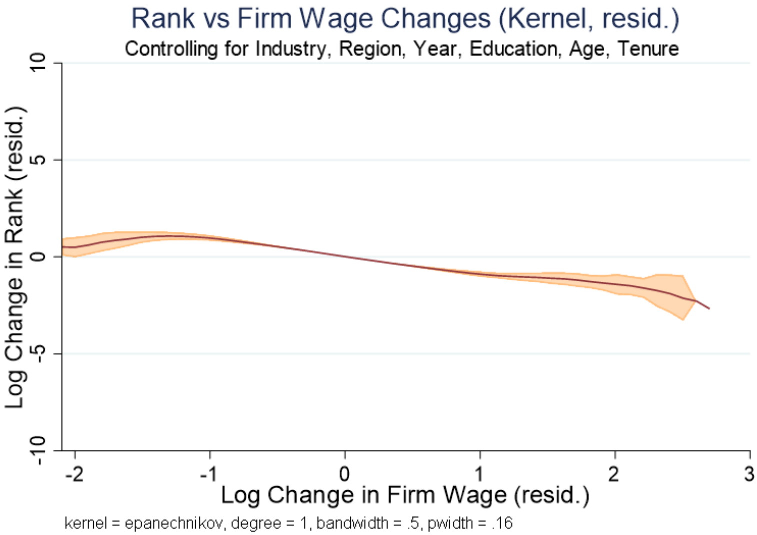
RESIDUALIZED BINSCATTER PLOTS



KERNEL REGRESSIONS - REJECT PAM ON LAYERS



KERNEL REGRESSIONS - EVIDENCE OF PAM ON RANKS



SUMMARY OF RESULTS

- Interesting sorting patters of workers and multi-worker firms
- Layers: If anything, better firms allocate workers in higher layers than worse firms
  - Better firm, the marginal contribution of each layer is smaller
  - NAM across layers
- Ranks: Better firms tend to allocate workers in lower ranks than worse firms
  - PAM across ranks
- Need of a framework that help us disentagle notions of layers and ranks within firms and rationalize these patterns

MODEL

MODEL SETUP

- Firms of type $z \sim G(z)$, workers of type $q \sim H(q)$
- Production organized in different layers/occupations indexed by n
- Firms choose number of layers (m) quality of worker in each layer (q_{nm})
- Workers choose (z, n) to maximize wages
- Number of layers determines the relative importance of each task for production

BASELINE MODEL

- Production function:

$$F\left(z, \{q_{nm}\}_{n=0}^{m}\right) = z \int_0^m a(n,m) q_{nm}^{\sigma} dn, \quad \sigma < 1 \tag{1}$$

- $a(n,m)$ is the relative importance of task n, $\frac{\partial a}{\partial n} > 0$, $\frac{\partial a}{\partial m} < 0$ ▸ Heuristic

FIRM'S PROBLEM

- The firm's problem consists of two stages
- Given structure choice, choose $\{q_{nm}\}_{n=0}^{m}$ to maximize operation profit:

$$\pi(z, m) = \max_{\{q_{nm}\}_{n=0}^{m}} z \int_0^m a(n, m) q_{nm}^{\sigma} dn - \int_0^m w(q_{nm}) dn \tag{2}$$

- When entering, choose m to maximize lifetime profit:

$$\max_{m} \pi(z, m) - c(m) \tag{3}$$

LABOR MARKET EQUILIBRIUM

- Main equilibrium objects: $w(q_{nm})$ and $\mu(z, n, m)$ ▸ Def
- FOC:

$$w'(q_{nm}) = \sigma z a(n, m) q_{nm}^{\sigma - 1}$$

- Guess (and later verify) $q_{nm} = \mu(n, m) z$

$$w(q_{nm}) = \frac{\sigma}{1 + \sigma} \frac{a(n, m) q_{nm}^{1+\sigma}}{\mu(n, m)}$$

LABOR MARKET EQUILIBRIUM

- Profit: ‣ Solution Algorithm

$$\pi(z,m) = \frac{z^{1+\sigma}}{1+\sigma}\int_0^m a(n,m)\mu(n,m)^{\sigma}dn$$

- To illustrate: assume $a(n,m) = \frac{a_0 n}{m}$, conjecture $\mu(n,m) = \frac{\mu_0 n}{m}$

STRUCTURE CHOICE

- Firm chooses m to maximize

$$\max_{m} \; \pi(z, m) - c(m)$$

- Assume $c(m) = \frac{\kappa m^2}{2}$, then FOC implies

$$m(z) = \beta z^{1+\sigma}$$

ALLOCATION

- Feasibility $\Rightarrow$ find $\mu_0 > 0$ that clears the market
- Allocation:

$$\mu(z, n, m) = \frac{\mu_0 n}{m(z)} z \Rightarrow \mu(z, n) = \frac{\mu_0 n}{\beta z^{\sigma}}$$

- NAM across firms (given *n*) comes from convexity in firm size
- Average worker inside firm $\bar{q}(z) \propto z^{2+\sigma}$ is increasing (PAM on average)

WAGES

- Plugging into $w(q_{nm})$:

$$w(z, n, m) = \frac{\sigma}{1+\sigma} \frac{a_0 \mu_0^{\sigma} n^{1+\sigma}}{m^{1+\sigma}} z^{1+\sigma}$$

- CEO pay grows exponentially:

$$w(z, m, m) = \frac{\sigma}{1+\sigma} a_0 \mu_0^{\sigma} z^{1+\sigma}$$

SUPERMODULARITY

- Reminder: a function $f(x, y)$ is supermodular if

$$f_{xy} > 0$$

- Sufficient condition for PAM in classical Becker (1973) model
- A function is log-supermodular if

$$\frac{f_{xy} f}{f_x f_y} > 1$$

- Strong condition, usually sufficient for PAM in search models (Shimer and Smith 2000, Eeckhout and Kircher 2011)

SUPERMODULARITY

- But production function is supermodular, shouldn't we expect PAM?
- We can write the production function as

$$F\left(z, \{q_{nm}\}_{n=0}^{m}\right) = \int_0^m a(n,m)\, f(z, q_{nm}) dn$$

- Supermodularity is not enough anymore. Now we need

$$\frac{f_{zq}(z,q)\, f(z,q)}{f_z(z,q)\, f_q(z,q)} > \frac{|\varepsilon_{a,m}||\varepsilon_{m,z}|}{\varepsilon_{f,z}}$$

- Potentially stronger than log-supermodularity!

SUPERMODULARITY

- Convexity in structure choice makes it so importance dominates gains in productivity
- Can show that under certain conditions this boils down to

$$|\varepsilon_{a,m}| < \frac{1}{1+\sigma}$$

▸ Proposition

- Idea: $a(n, m)$ is "concave enough"

COMPARING HIERARCHIES

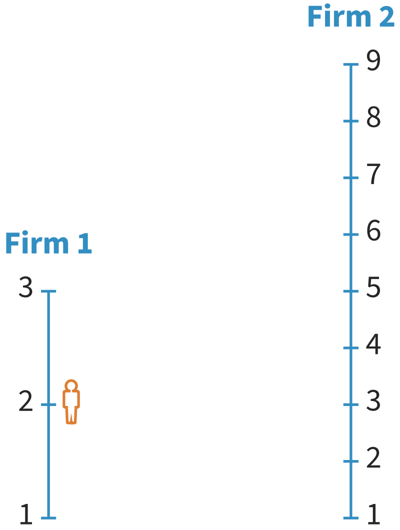
- When would two firms hire the same worker?

$$\mu(z,n) = \mu(z',n') \Leftrightarrow \frac{n'}{n} = \left(\frac{z'}{z}\right)^{\sigma}$$

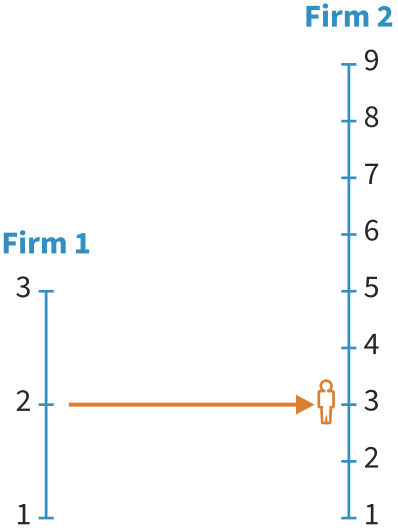
- Define $\varphi = \frac{n}{m}$ (worker rank in the firm):

$$\mu(z,n,m) = \mu(z',n',m') \Leftrightarrow \frac{\varphi}{\varphi'} = \frac{z'}{z}$$

COMPARING HIERARCHIES



COMPARING HIERARCHIES



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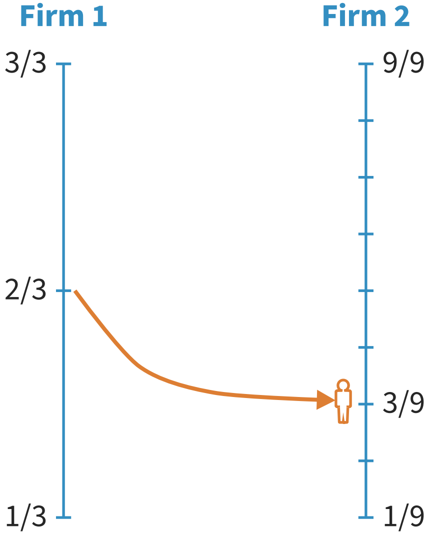


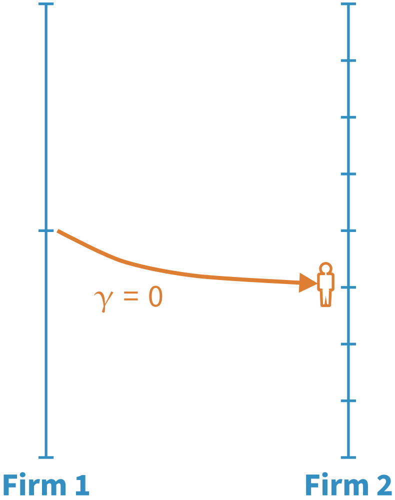
COMPARING HIERARCHIES: DISCUSSION

- Recontextualizes job transitions
- What is the correct notion of PAM in this case?
- Suppose we have peer effects (similar to Manski's reflection problem in logs):

$$f(z, q_{nm}) = z \left(\int_0^m q_{nm} dn \right)^{\gamma} q_{nm}^{\sigma}$$

  - Stronger peer effects ($\uparrow \gamma$) exacerbate "PAM-ness" in ranks

COMPARING HIERARCHIES (PEER EFFECTS)



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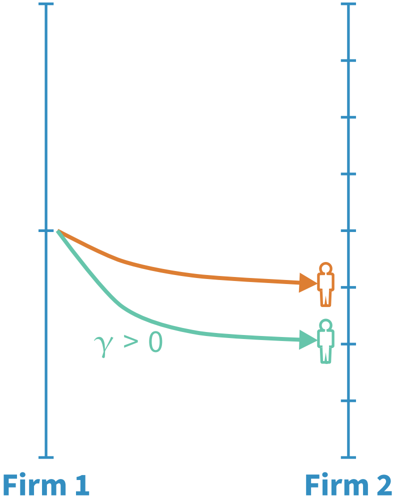

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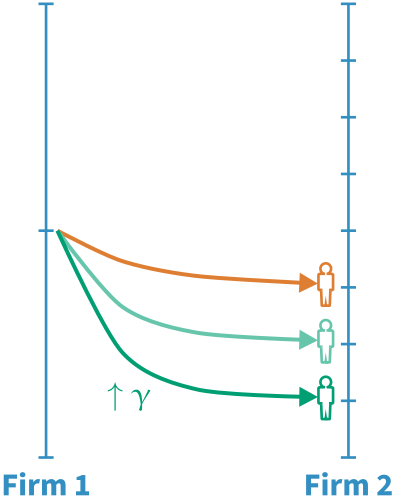


POWER LAWS

- Suppose z follows a power law

$$Prob(z > x) \sim S(x)x^{1-\eta}, \quad \eta > 1$$

Proposition

If z follows a power law, then $m(z)$ also follows a power law. In particular, if z is Pareto with scale parameter z_{min} and shape parameter $\eta > 1 + \sigma$, then $m(z)$ is Pareto with scale $\beta z_{min}^{1+\sigma}$ and shape $\frac{\eta}{1+\sigma}$.

- Under Pareto assumption, can recover distribution of z from hierarchies.

POWER LAWS

- Important fact from wage dispersion literature: right tail of distribution of log wages is convex

Proposition

If the right tail of q follows a power law with $S(x) = s$, then the distribution of log wages is convex

- Disciplines the types of distributions that are reasonable for q

WAGE DISPERSION

- Maintaining Pareto assumption, we can compute within-firm log wage dispersion:

$$V^{within} = \mathbb{E}\left[Var\left(\log w|z\right)\right]$$

- Between-firm log wage dispersion:

$$V^{between} = Var\left(\mathbb{E}\left[\log w|z\right]\right)$$

- Use the model to target relative sizes of within vs. between firm log wage dispersion
- Illustration: Song et al. (2019) estimate a ratio of $\frac{V^{within}}{V^{between}} \approx 1.5$

WAGE DISPERSION

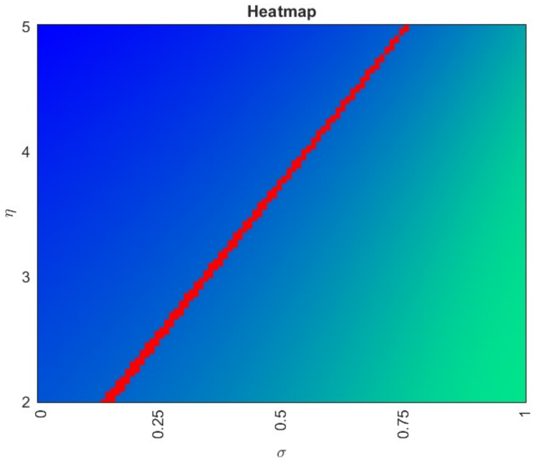
$\kappa = 0.1$

$z_{min} = 1$

Green is high

Blue is low

Red means $\frac{V^{within}}{V^{between}} \approx 1.5$



$a_0 = 1$

- Aggregate output

$$Y = \int F\left(z, \{\mu(z,n,m)\}_{n=0}^{m}\right) dG(z)$$

Proposition

Suppose z follows a Pareto distribution with minimum value z_{min} and the production function follows (1). Then $Y = AL$, where L is the total effective units of labor in the economy and A is a constant.

EXTENSION: DIRECTED SEARCH

- Suppose firms post wages and workers directedly search for jobs
- Define $\lambda(z, n, m)$ as the tightness ratio and $p(\lambda)$ is the prob of finding a job
- Firms solve

$$\max \int_0^m p(\lambda(z, n, m)) \left(a(n, m) f(z, q_{nm}) - w(q_{nm})\right)$$

- Workers solve

$$\max_{(z,n,m)} \frac{p(\lambda(z, n, m))}{\lambda(z, n, m)} w(q_{nm})$$

EXTENSION: DIRECTED SEARCH

- Similar environment to Eeckhout and Kircher (2011)
- However, supermodularity requirement is potentially even stronger now

$$\frac{f_{zq}(z,q)\, f(z,q)}{f_z(z,q)\, f_q(z,q)} > \frac{|\varepsilon_{a,m}||\varepsilon_{m,z}| + |\varepsilon_{p,\lambda}||\varepsilon_{\lambda,z}|}{\varepsilon_{f,z}}$$

- Can't find conditions on primitives here yet, but may be promising

FINAL REMARKS

- Empirical evidence of rich patterns of sorting between workers and multi-worker firms
- Constructed a sorting model with multi-worker firms and structure choice has empirically consistentconsequences for sorting
- The importance function + assignment function give us a way of comparing workers/occupations across firms of different productivities

NEXT STEPS

- Characterize the directed search extension and quantitative analysis
- Provide an estimate for the importance and assignment function using model and data
- Explore implications for firm dynamics and growth
- Counterfactuals: …

Thank You!

fllopes@wisc.edu

gstoledo@purdue.edu

REFERENCES

- Adenbaum, Jacob (2022). "Endogenous firm structure and worker specialization". PhD thesis. University of Minnesota.
- Boerma, Job, Aleh Tsyvinski, and Alexander P Zimin (2021). *Sorting with team formation*. Tech. rep. National Bureau of Economic Research.
- Dauth, Wolfgang and Johann Eppelsheimer (2020). "Preparing the sample of integrated labour market biographies (SIAB) for scientific analysis: a guide". In: *Journal for Labour Market Research* 54(1), pp. 1–14.
- Deming, David J (2017). "The growing importance of social skills in the labor market". In: *The Quarterly Journal of Economics* 132(4), pp. 1593–1640.

REFERENCES II

Eeckhout, Jan and Philipp Kircher (2018). "Assortative matching with large firms". In: *Econometrica* 86(1), pp. 85–132.

Freund, Lukas (2022). "Superstar Teams: The Micro Origins and Macro Implications of Coworker Complementarities". In: *Available at SSRN 4312245*.

Garicano, Luis (2000). "Hierarchies and the Organization of Knowledge in Production". In: *Journal of Political Economy* 108(5), pp. 874–904.

Garicano, Luis and Esteban Rossi-Hansberg (2006). "Organization and inequality in a knowledge economy". In: *The Quarterly Journal of Economics* 121(4), pp. 1383–1435.

Gregory, Victoria (2020). "Firms as learning environments: Implications for earnings dynamics and job search". In: *FRB St. Louis Working Paper*(2020-036).

REFERENCES III

- Jarosch, Gregor, Ezra Oberfield, and Esteban Rossi-Hansberg (2021). “Learning from coworkers”. In: *Econometrica* 89(2), pp. 647–676.
- Kremer, Michael (1993). “The O-ring theory of economic development”. In: *The Quarterly Journal of Economics* 108(3), pp. 551–575.
- Kremer, Michael and Eric Maskin (1996). *Wage inequality and segregation by skill*.

Appendix

MICROFOUNDATION

- Law firm with two workers, lawyer and paralegal, each with 1/3 units of time
- Cases have a random "profile" $x \sim \mathcal{U}\,[0, 1]$
- The firm chooses whether to accept the cases and to whom to assign the case
- Solution: accept cases with profile $x \geq 1/3$
- Assign cases $x \in [1/3, 2/3]$ to paralegal, $x \geq 2/3$ to lawyer
- Avg productivity of paralegal = 1/2, avg productivity of lawyer = 5/6

▸ Back

MICROFOUNDATION

- Now firm has hired a senior lawyer
- What happens to the others' participation in total revenue?
- New solution: accept all cases, $x \leq 1/3$ to paralegal, $x \in [1/3, 2/3]$ to lawyer, $x \geq 2/3$ to senior lawyer
- Paralegal participation: $1/2 \to 1/6$, lawyer participation: $5/6 \to 1/2$
- Firm hires senior lawyer if additional caseload is worth it

▸ Back

DEFINITION OF EQUILIBRIUM

Definition

A competitive equilibrium is a set of functions $\mu : \mathcal{Z} \times \mathbb{R} \times \mathbb{R} \to \mathcal{Q}$, $w : \mathbb{Q} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $m : \mathcal{Z} \to \mathbb{R}$ such that

i. μ solves (2)

ii. m satisfies (3)

iii. w satisfies worker optimality condition:

$$w(q,n,m) \geq w(q,n',m'),\ \forall (n',m') \text{ s.t. } n' \leq m' = m(z), \text{ for some } z \in \mathcal{Z}$$

iv. the allocation is feasible:

$$\int_{z}^{\bar{z}} \int_{0}^{m(z)} dndG(z') = \int_{0}^{m(z)} \int_{\mu(z,n,m)}^{\bar{q}} dH(q')dn,\ \forall z \in \mathcal{Z}$$

Back

SOLUTION ALGORITHM

- Define $\varphi \equiv \frac{n}{m}$
- Rewrite maximization problem independent from m

$$\max_{\{q_\varphi\}_{\varphi=0}^{1}} \int_0^1 a(\varphi)\, f(z, q_\varphi) d\varphi - \int_0^1 w(q_\varphi)$$

- FOC implies

$$w'(\mu(z,\varphi)) = a(\varphi)\, f_q(z, \mu(z,\varphi))$$

SOLUTION ALGORITHM

- Feasibility:

$$\int_{z}^{\bar{z}} \int_{0}^{\varphi} g(z')d\varphi' dz' = \int_{\mu(z,\varphi)}^{\bar{q}} h(q')dq', \;\; \forall z$$

- This implies the following differential equation

$$\frac{\partial \mu(z,\varphi)}{\partial z} = \varphi \frac{g(z)}{h(\mu(z,\varphi))}$$

SOLUTION ALGORITHM

- Then, labor market equilibrium is given by a family of functions $\{w_\varphi, \mu_\varphi\}_{\varphi=0}^{1}$ that satisfy the following system of differential equations:

$$\frac{dw_\varphi(z)}{dz} = a(\varphi)\, f_q(z, \mu_\varphi(z)) \frac{d\mu_\varphi}{dz}(z)$$
$$\frac{d\mu_\varphi(z)}{dz} = \varphi \frac{g(z)}{h(\mu_\varphi(z))}$$

▸ Back

SUPERMODULARITY PROPOSITION

Proposition

Suppose $a(n, m) = \tilde{a}(\frac{n}{m})$, where $\tilde{a}(\cdot)$ is a polynomial that takes $\frac{n}{m}$ as its argument. Then, the allocation exhibits PAM if

$$\left|\varepsilon_{a,m}\right| < \frac{1}{1 + \sigma}$$

Back